GRAPH THEORY – EXAM

05 JANUARY 2023

Please note: This exam consists of 5 problems, worth 8p each. In order to pass this part of the course examination, you will need to obtain at least 18 points out of 40.

For solving this exam, no aids are allowed. When using colours, please abstain from using the colour red.

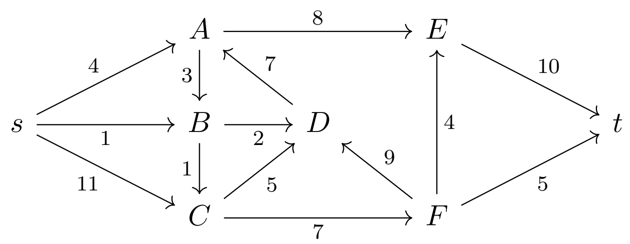
Unless stated otherwise, all graphs in this exam are assumed to be finite simple graphs.

– *Good luck!*

Problem 1. For each of the following statements, decide whether they are right or wrong, and prove (or disprove) them. (2p each)

(a) A finite simple graph G has a unique spanning tree if and only if G is itself a tree.
(b) A Hamiltonian graph on an even number of vertices has at least two perfect matchings.
(c) Any 3-regular graph has chromatic number at least 3.
(d) The Petersen graph has a path through all 10 vertices.

Problem 2. Consider the following flow network:



(a) Employ the Ford-Fulkerson algorithm to find a maximum flow. Your solution should contain all relevant steps of the algorithm. (4p)
(b) Find a minimum s-t-cut in the above network. (1p)
(c) Now forget about the directions of the arrows, and regard the capacities as edge weights on the resulting undirected graph instead. Find the distances between s and any other vertex using Dijkstra's algorithm. Your solution should contain all relevant steps of the algorithm. (3p)

Problem 3. (a) State and prove Dirac's theorem about the existence of Hamilton cycles. (4p)
(b) State (one version of) Menger's theorem. (2p)
(c) Construct a sequence of finite simple graphs $G_n = (V_n, E_n)$ such that $|V_n| = n$, $\kappa(G_n) = 1$, and $\lim_{n\to\infty} \binom{n}{2}^{-1}|E_n| = 1$. (2p)

Problem 4. Let $n \geq 3$. Construct a finite simple graph G_n in the following way: Start with 2 copies of C_n, where the vertices are counterclockwise labelled $v_1, \ldots, v_n$ and $w_1, \ldots, w_n$, respectively. Then draw additional edges between w_i and all the neighbours of v_i, for $i = 1, \ldots, n$.

(a) Show that G_3 is planar. (1p)
(b) Show that G_n is nonplanar for $n \geq 4$. (2p)
(c) Show that G_n is Hamiltonian for $n \geq 3$. (2p)
(d) Show that G_n has the chromatic index $\chi'(G_n) = 4$ for $n \geq 3$. (3p)

Problem 5. Consider the random graph $G(n, p)$.

(a) What does it mean for a function $f(n)$ to be a threshold function for a property (P) of $G(n, p)$? (2p)
(b) Denote by X the number of connected components that are copies of K_2 in $G(n, p)$. Determine $\mathbf{E}[X]$. (2p)
(c) Determine $\mathbf{E}[X^2]$. (3p)
(d) Is there a threshold function for the occurrence of connected components that are copies of K_2 in $G(n, p)$? (1p)